Discussion Paper 7923 December 1979

LARGE SAMPLE ASYMPTOTIC EXPANSIONS FOR GENERAL LINEAR SIMULTANEOUS SYSTEMS UNDER MISSPECIFICATION

Roberto S. Mariano
John G. Ramage

ABSTRACT

Based on large-sample stochastic approximations, we obtain asymptotic bias and mean squared error for the k-class estimators of an identified but misspecified equation with an arbitrary number of endogenous variables in a linear simultaneous-equations model. The specification error considered here is the omission of appropriate exogenous variables from the equation being estimated.

This paper extends existing results in various respects. It treats not only consistent estimators but also inconsistent ones, like ordinary least squares, and it provides conditions under which ordinary least squares will be preferred among those estimators under study. It covers cases where the estimated equation contains three or more endogenous variables. It includes the limited-information maximum likelihood as one of the estimators under analysis. It also contains a rigorous argument for the propriety of the stochastic approximations from which asymptotic moments of estimators are calculated.

LARGE SAMPLE ASYMPTOTIC EXPANSIONS FOR GENERAL LINEAR SIMULTANEOUS SYSTEMS UNDER MISSPECIFICATION*

Roberto S. Mariano
*John G. Ramage***

December, 1979

I. INTRODUCTION AND SUMMARY

Through large-sample stochastic approximations, the effects of misspecification are analyzed for single-equation estimators (k-class, including limited-information maximum likelihood, and modified two-stage least squares) of an identified equation with an arbitrary number of endogenous variables in a linear simultaneous equations model. The focus is on the omission of appropriate exogenous variables from the equation being estimated.

This paper extends Fisher's results (1961, 1966, 1967) in that inconsistent estimators like ordinary least squares (OLS) are considered as well and conditions are given under which OLS will be preferred among those estimators under study. The paper extends

---

*For presentation in the annual meeting of the Econometric Society to be held in Atlanta, Georgia on December 28-30, 1979.

**The authors gratefully acknowledge support from NSF Grant SOC79-07964. The first author also acknowledges the hospitality of the School of Economics, University of the Philippines, where he is visiting for November-December, 1979.

Mariano and Ramage (November, 1978) to cover cases where the estimated equation contains three or more endogenous variables. The approach taken, though similar to Nagar's (1959) where large-sample asymptotic expansions are developed for the estimators themselves, provides various refinements of the technique thereby showing the usefulness and wider applicability of the approach for finite-sample analysis. The propriety of the approximations is rigorously shown: the deleted remainder in the expansion has the correct (small) order of magnitude in probability. Furthermore, a simple modification allows us to apply the technique to k-class estimators not only of the form $k = 1 + (b/N)$ (N = sample size, b a fixed constant) as in Nagar, but also to other values of k, especially $0 \leq k < 1$ and k stochastic.

The results are specialized to the case of two endogenous variables to further illustrate the conditions under which ordinary least squares will dominate the k-class in terms of asymptotic mean squared error.

2. ASYMPTOTIC EXPANSIONS IN THE GENERAL CASE

Consider a simultaneous system of linear stochastic equations and assume that the equation to be estimated has the following forms:

$$\begin{aligned} &\text{True} \quad : \; y_1 = Y_2\beta + Z_1\gamma_1 + Z_2{}^*\gamma_2 + u^* \;, \\ &\text{Apparent:} \; y_1 = Y_2\beta + Z_1\gamma_1 + u \;, \end{aligned} \tag{2.1}$$

where y_1, Y_2, Z_1, Z_2^*, u^* and u are $N\times 1$, $N\times G_1$, $N\times K_1$, $N\times K_2^*$, $N\times 1$ and $N\times 1$ respectively. The first two symbols refer to endogenous variables, the next two to exogenous variables and the last two to structural disturbances.

Denote the exogenous variables in the whole system by

$$\text{True} \quad : \quad Z^* = (Z_1 \; Z_2^* \; Z_3^*) \; : N\times K^*, \; K^* = K_1 + K_2^* + K_3^*$$

$$\text{Apparent} \; : \quad Z = (Z_1 \; Z_2 \; Z_3) \quad : N\times K \;, \; K = K_1 + k_2 + k_3 \tag{2.2}$$

Under classical assumptions, there are no lagged endogenous variables in the system and from the reduced form equations we get

$$Y_2 = M_* + V_* = Z^*\Pi_2 + V \tag{2.3}$$

where the N rows of V are mutually independent and identically distributed as multivariate normal with mean zero and covariance matrix Σ $(G_1 \times G_1)$.

The k-class estimator of β based on the apparent equation in (2.1) is

$$\hat{\beta}_{(k)} = \beta + (Y_2'P_kY_2)^{-1} \; Y_2'P_k(Z_2^*\gamma_2 + u^*) \tag{2.4}$$

where

$$P_k = k(P_Z - P_{Z_1}) + \bar{k}\,\bar{P}_{Z_1}$$

$$= \bar{k}\,\bar{P}_z + (P_z - P_{z_1}) \tag{2.5}$$

$$= -k\,\bar{P}_z + \bar{P}_{z_1}$$

and

$$P_z = Z(Z'Z)^{-1} Z'$$

$$\bar{P}_z = 1 - P_z \quad . \tag{2.6}$$

If we let Q be a nonsingular $(G_1 \times G_1)$ matrix such that

$$Q' \Sigma Q = I \tag{2.7}$$

and define

$$M = M_* Q \quad , \quad V = V_* Q \quad , \quad u = u^*/\omega \quad , \tag{2.8}$$

where $\omega^2 = E(u_t^*)^2$, then, from (2.4) we can write

$$\hat{\beta}_{(k)} - \beta = QA^{-1} B$$

$$= \omega Q(I + A_1^{-1} A_{1/2} + A_1^{-1} A_0)^{-1} A_1^{-1} (B_1 + B_{1/2} + B_0) \tag{2.9}$$

where

$$A = (M+V)' P_k (M+V) \tag{2.10}$$

$$B = (M+V)' P_k (Z_2^* \gamma_2/\omega + u)$$

$$= (M+V)' P_k (\eta + u) \tag{2.11}$$

for

$$\eta = \bar{P}_{z_1} Z_2^* \gamma_2/\omega \quad . \tag{2.12}$$

Note that the second equality in (2.11) holds since $P_k \bar{P}_{z_1} = P_k$. To identify components of various orders of magntiude (in powers of N), we can write A and B as

$$A = A_0 + A_{1/2} + A_1$$
$$B = B_0 + B_{1/2} + B_1 \tag{2.13}$$

where, for

$$\bar{k} = 1-k \quad ; \quad \zeta = E(v_t \ u_t^*)/\omega \quad , \tag{2.14}$$

$$A_1 = M'P_kM + \bar{k}(N-K)I$$
$$A_{1/2} = \bar{k}[V'\bar{P}_zV - (N-k)I] + V'P_kM + M'P_kV$$
$$A_0 = V'P_1V \tag{2.15}$$
$$B_1 = M'P_k\eta + \bar{k}(N-K)\zeta$$
$$B_{1/2} = M'P_ku + \bar{k}[V'\bar{P}_zu - (N-K)\zeta] + V'P_k\eta$$
$$B_0 = V'P_1u \quad .$$

The subscripts in the above components indicate their orders of magnitude. Thus $A_1 = O(N)$, $A_{1/2} = O_p(N^{1/2})$, etc.

The asymptotic expansion and approximation to $\hat{\beta}_{(k)} - \beta$ is obtained from the infinite series representation of the matrix inverse in (2.9). The rationale comes from the following:

Proposition 1. If $(Z^{*\prime}Z^*)/N$ converges to a finite positive definite limit as $N \to \infty$, then, under the set of assumptions made earlier about the model, for k nonstochastic and for any nonnegative integer r,

$$\hat{\beta}_{(k)} - \beta = Q \sum_{j=0}^{r} \{(-1)^j [A_1^{-1}(A_{1/2}+A_0)]^j A_1^{-1}(B_1+B_{1/2}+B_0)$$

$$+ o_p[(N^{-(r+1)/2}]\} . \qquad (2.16)$$

Proof. See Ramage (1971), Lemmas A-1.6 through Lemmas A-1.9.

Note that the expansion in Proposition 1 is analogous to Nagar (1959). Here, however, we have provided a formal framework within which the expansion can be interpreted as a stochastic approximation to the estimation error. Furthermore, the expansion is valid for any nonstochastic k and not only for values of k considered by Nagar, namely, $k = 1 + b/N$.

In the case where k is stochastic, we need a modification of Proposition I. If $k = O_p(1)$, the indicated orders of magitude in (2.15) are still applicable. However, $A_{1/2}$, A_1, $B_{1/2}$ and B_1 contain terms of lower orders of magnitude. Suppose the following expansion for k holds:

$$k = k_0 + k_{-1/2} + k_{-1} + k_r \qquad (2.17)$$

where $k_r = O_p(N^{-3/2})$, $k_0 = O(1)$, $k_{-1/2} = O_p(N^{-1/2})$ and $k_{-1} = O_p(N^{-1})$. Then terms of smaller order in the components of A and B would come from those having $k_{-1/2}$ or k_{-1} as a factor.

The appropriate grouping of terms, with (2.17) holding, is, for A,

$$A = \tilde{A}_1 + \tilde{A}_{1/2} + \tilde{A}_0 + R_a \quad , \tag{2.18}$$

where

$$\begin{aligned}
\tilde{A}_1 &= A_1(k_0) = M'P_{k_0}M + \bar{k}_0(N-K)I \quad , \\
\tilde{A}_{1/2} &= A_{1/2}(k_0) - k_{-1/2}[M'\bar{P}_zM + (N-K)I] \quad , \\
\tilde{A}_0 &= A_0 - k_{-1}[M'\bar{P}_zM + (N-K)I] \\
&\quad -k_{-1/2}[V'\bar{P}_zV - (N-k)I + V'\bar{P}_zM + M'\bar{P}_zV] \quad , \\
R_a &= O_p(N^{-1/2}) \quad ;
\end{aligned} \tag{2.19}$$

and, for B,

$$B = \tilde{B}_1 + \tilde{B}_{1/2} + \tilde{B}_0 + R_b \quad , \tag{2.20}$$

where

$$\begin{aligned}
\tilde{B}_1 &= B_1(k_0) = M'P_{k_0}\eta + \bar{k}_0(N-K)\zeta \quad , \\
\tilde{B}_{1/2} &= B_{1/2}(k_0) - k_{-1/2}[M'\bar{P}_z\eta + (N-K)\zeta] \\
\tilde{B}_0 &= B_0 - k_{-1}[M'\bar{P}_z\eta + (N-K)\zeta] \\
&\quad -k_{-1/2}[M'\bar{P}_zu + (V'\bar{P}_zu - (N-K)\zeta) + V'\bar{P}_z\eta] \\
R_b &= O_p(N^{-1/2})
\end{aligned} \tag{2.21}$$

Analogous to Proposition 1, we then have

Proposition 2. For k stochastic, in addition to the assumptions in Proposition 1, suppose (2.17) holds. Then, for any nonnegative integer r,

$$\hat{\beta}_{(k)}-\beta = \omega Q \sum_{j=0}^{r} \{(-1)^j [\tilde{A}_1^{-1}(\tilde{A}_{1/2}+\tilde{A}_0+R_a)]^j \tilde{A}^{-1}(\tilde{B}_1+\tilde{B}_{1/2}+\tilde{B}_0+R_b)\}$$
$$+ o_p[N^{-(r+1)/2}] , \qquad (2.2)$$

where R_a and R_b are remainder terms, both of order $N^{-1/2}$ in probability.

Note that in addition to the obvious difference between the nonwiggled and wiggled components of A and B, terms of order $N^{-3/2}$ and higher negative powers of N in the expansions under stochastic and nonstochastic k will also differ because of R_a and R_b.

3. ASYMPTOTIC MOMENTS

The leading terms in the expansions in Propositions 1 and 2 are:

Corollary 1.

$$\hat{\beta}_{(k)}-\beta = \omega QF + o_p(N^{-3/2}) \qquad (3.1)$$

where, for k nonstochastic,

$$F = F_0 + F_{-1/2} + F_{-1}$$

$$F_0 = A_1^{-1} B_1$$

$$F_{-1/2} = A_1^{-1}(B_{1/2} - A_{1/2}A_1^{-1} B_1) \tag{3.2}$$

$$F_{-1} = A_1^{-1}[B_0 - A_0 A_1^{-1} B_1 - A_{1/2}A_1^{-1} B_{1/2} + (A_{1/2}A_1^{-1})^2 B_1]$$

For k stochastic and satisfying (2.17), the expressions for F_0, $F_{-1/2}$ and F_{-1} are the same, except for the presence of wiggles on the A and B components.

Up to order N^{-1}, ωQF is then a stochastic approximation to $(\hat{\beta}_{(k)}-\beta)$. For this reason, we will then refer to the moments of ωQF as the asymptotic moments of $(\hat{\beta}_{(k)}-\beta)$.

Note from (3.2) that F_0 is nonstochastic. Also, since A_1 and B_1 are nonstochastic and $E(B_{1/2})=0$ and $E(A_{1/2})=0$, it follows then that $E(F_{-1/2})=0$ and $E(F_0 F'_{-1/2})=0$. Thus, denoting the asymptotic bias and asymptotic mean squared error of $\hat{\beta}_{(k)}$ by $\mathcal{B}(k)$ and $M(k)$, we get for k nonstochastic:

$$\mathcal{B}(k) = \omega QE(F) = \mathcal{B}_0 + \mathcal{B}_{-1/2} + \mathcal{B}_{-1}$$

where

$$\mathcal{B}_0 = F_0 = A_1^{-1} B_1$$

$$\mathcal{B}_{-1/2} = E(F_{-1/2}) = 0$$

$$\mathcal{B}_{-1} = E(F_{-1}) \quad .$$

For asymptotic mean squared error,

$$M(k) = \omega^2 QE(FF')Q'$$

$$= \omega^2 Q[M_0 + M_{-1/2} + M_{-1} + M_{-3/2}]Q'$$

where

$$M_0 = F_0'F_0 = \mathcal{B}_0\mathcal{B}_0'$$

$$M_{-1/2} = E(F_0F'_{-1/2} + F_{-1/2}F_0') = 0 \tag{3.4}$$

$$M_{-1} = B_0B'_{-1} + B_{-1}'B_0 + E(F_{-1/2}F'_{-1/2})$$

$$M_{-3/2} = E(F_{-1}F'_{-1/2} + F_{-1/2}F'_{-1}) \quad .$$

For k nonstochastic, $\mathcal{B}_{-1}$ and M_{-1} have been evaluated but they need not be reported here. Following our earlier convention, let us also use wiggles to denote components pertaining to stochastic k satisfying (2.17). For k stochastic, the first leading terms in the moment expression would apply still, if evaluated at k_0. However, succeeding terms will have to be modified: indeed, $\tilde{\mathcal{B}}_{-1/2}$ will not equal zero, in general, and $\tilde{\mathcal{B}}_{-1}$ will certainly differ from $\mathcal{B}_{-1}$.

We end this section with some comments concerning the determination of k_0 in (2.17) for the limited information maximum likelihood estimator. For the general case (arbitrary value for G_1 in (2.11)), we have no closed form expression for k_0 for LIML. However, a method of calculating it is as follows.

The LIML estimator can be represented in the form of (2.4) for

$$k = 1 + \ell \tag{3.5}$$

where ℓ is the smallest root of the determinantal equation, for $Y = (y_1 \ Y_2)$,

$$|Y'P_1Y - \ell Y'\bar{P}_zY| = 0 \tag{3.6}$$

A closed form expression can be easily obtained when Y is 2x2 (i.e., $G_1=1$) but not for arbitrary G_1. However, it can be shown for arbitrary G_1 that

$$\ell = O_p(1) \quad \text{as} \quad N \to \infty \tag{3.7}$$

and if

$$\ell = \ell_0 + O_p(N^{-1/2}) \ , \tag{3.8}$$

then we would have

$$\ell_0 = C_1/D_1 \ . \tag{3.9}$$

In (3.9), C_1 and D_1 are determined from the following expressions

$$\begin{aligned} \hat{\beta}_*'Y'\bar{P}_{z_1}Y\hat{\beta}_* &= C_1 + C_{1/2} + O_p(1) \\ \hat{\beta}_*'Y'\bar{P}_z Y\hat{\beta}_* &= D_1 + D_{1/2} + O_p(1) \end{aligned} \tag{3.10}$$

where $\hat{\beta}_*' = (-1, \hat{\beta}_{LIML})$. Both C_1 and D_1 contain ℓ_0 and thus (3.9) defines ℓ_0 only implicitly.

4. THE CASE OF TWO ENDOGENOUS VARIABLES

To simplify discussion in this section, assume that Z^* is a column submatrix of Z. We are thus limiting the discussion to what Hale, Mariano and Ramage (1979) refer to as Type 1 specification error. Under this assumption, $\bar{P}_z M=0$ and the leading term in (3.2) which is nonstochastic and $O(1)$ simplifies to

$$F_0 = A_1^{-1}B_1 = \zeta - (C + \bar{k}I)^{-1}C(\zeta - C^{-1}D) \tag{4.1}$$

where

$$\begin{aligned} C &= M'\bar{P}_{z_1}M/(N-K) \\ D &= M'\eta/(N-K) = M'\bar{P}_{z_1}Z_2{}^*\gamma_2/[\omega(N-K)] \quad . \end{aligned} \tag{4.2}$$

and ζ is defined in (2.14)

For the case $G_1 = 1$, the above expression for F_0 further simplifies to that given in Mariano and Ramage (1978):

$$F_0 = R - \alpha_k(R - \theta/\delta) \tag{4.3}$$

where

$$R = \rho\omega/\sigma$$

$$\sigma^2 = \mathrm{Var}\ (y_{t2})$$

$$\rho = E(y_{t2}u_t{}^*)/(\omega\sigma)$$

$$\theta = \pi_2{}'Z^*{}'\bar{P}_{Z_1} Z_2^*{}'\gamma_2/(2\sigma^2) \tag{4.4}$$

$$\delta = \pi_2{}'Z^*{}'\bar{P}_{Z_1} Z^*\pi_2/(2\sigma^2)$$

$$\alpha_k = \delta/[\delta + \bar{k}(N-K)/2] \quad .$$

In this case, also, a closed form expression can be obtained for ℓ_0, see Mariano and Ramage (1978):

$$\ell_o = [c - (c^2 - 4\,a\,b)^{1/2}]/(2b) \tag{4.5}$$

so that for LIML, $k_0 = 1 + \ell_0$.

In (4.5), we have

$$\begin{aligned} a &= (\nu/\delta) - (\theta/\delta)^2 \\ b &= [(N-K)/(2\delta)]^2 \; (1-\rho^2) \; (\sigma/\omega) \\ c &= [(N-K)/(2\delta)] \; [(1-\rho^2)(\sigma/\omega) + R^2 - 2R(\theta/\delta) + (\nu/\delta)] \end{aligned} \tag{4.6}$$

$$\nu = \gamma_2{}'Z_2^*{}'\bar{P}_{Z_1} Z_2^*\gamma_2/(2\sigma^2) \quad .$$

Note that (4.3) describes a rectangular hyperbolic relationship between F_0 and k. This can then be exploited to determine the dominant member of the k-class for various regions of the parameter space. See Mariano and Ramage (1978).

REFERENCES

Fisher, F.M. (1961). "On the Cost of Approximate Specification in Simultaneous Equation Estimation." Econometrica 29, pp. 139-170.

_______ (1966). "The Relative Sensitivity to Specification Error of Different k-class Estimators." JASA 61, pp. 345-356.

_______ (1967). "Approximate Specification and the Choice of a k-class Estimator." JASA 62, pp. 1265-1276.

Hale, C., R.S. Mariano and J.G. Ramage (1978). "Finite Sample Analysis of Misspecification in Simultaneous Equation Models." JASA, forthcoming.

Mariano, R.S. and J.G. Ramage (1978). Ordinary Least Squares Versus Other Single-Equation Estimators: A Return Bout Under Misspecification in Sumultaneous Systems." Discussion Paper # 400, Department of Economics, University of Pennsylvania.

Nagar, A.L. (1959). "The Bias and Moment Matrix of the General k-class Estimators of Parameters in Simultaneous Equations." Econometric 27, pp. 575-595.

Ramage, J.G. (1971). "A Perturbation Study of the k-class Estimators in the Presence of Specification Error." Unpublished Ph.D. Thesis, Yale University.